Basic Linear Algebra

Alphabet

I	**Identity Matrix**
$\det(A)$	**Determinant** of Matrix A
$\operatorname{adj}(A)$	**Adjoint** of Matrix A
C_{ij}	**Cofactor** of Entry a_{ij}
$\lVert\mathbf{v}\rVert$	**Norm** of $\mathbf{v}$
$\mathbb{R}^n$	The set of all ordered real n-tuples
$\mathbb{C}^n$	The set of all ordered complex n-tuples
$[\mathbf{v}]_B$	Coordinate of $\mathbf{v}$ relative to Basis B
$P_{B\to B'}$	**Transition Matrix** from B to B'
$\operatorname{span}(S)$	Span of Vector Set S
D	**Diagonal Matrix**
Q	**Orthogonal Matrix**
A^*	**Conjugate Transpose** of Matrix A

Elementary Row Operations

1. **Multiply** a row through by a nonzero constant.
2. **Interchange** two rows.
3. **Add** a constant times one row to another.

Gaussian Elimination

$$\underset{\text{Row Echelon}}{\begin{bmatrix}1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & *\end{bmatrix}} \longrightarrow \underset{\text{Reduced Row Echelon}}{\begin{bmatrix}1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & *\end{bmatrix}}$$

Find Inverse of Matrix

$$\left[A \mid I\right] \xrightarrow{\text{row operations}} \left[I \mid A^{-1}\right]$$

LU-Decomposition

$$A \xrightarrow[\text{without row interchanges}]{\text{Gaussian Elimination}} LU$$

Determinants signed n-dimension volume

Cofactor Expansion

$$\begin{vmatrix}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = a_{11}\begin{vmatrix}a_{22} & a_{23} \\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix}a_{21} & a_{23} \\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix}a_{21} & a_{22} \\ a_{31} & a_{32}\end{vmatrix}$$

Row Reduction

$$\begin{vmatrix}ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = k\begin{vmatrix}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$\begin{vmatrix}a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = -\begin{vmatrix}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$\begin{vmatrix}a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = \begin{vmatrix}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

Properties

$$\begin{vmatrix}a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} = \begin{vmatrix}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix} + \begin{vmatrix}b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$$\begin{vmatrix}a_{11} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ & & a_{33}\end{vmatrix} = a_{11}a_{22}a_{33}$$

$$\det(A) = \det(A^T)$$
$$\det(A)\det(A^{-1}) = 1$$
$$\det(AB) = \det(A)\det(B)$$

Adjoint

$$adj(A) = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

$$A\ adj(A) = \det(A)I$$

Cramer's Rule - the solution of $A\mathbf{x} = \mathbf{b}$

$$x_i = \frac{\begin{vmatrix} a_{11} & \dots & a_{1(i-1)} & b_1 & a_{1(i+1)} & \dots & a_{1n} \\ a_{21} & \dots & a_{2(i-1)} & b_2 & a_{2(i+1)} & \dots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \dots & a_{n(i-1)} & b_1 & a_{n(i+1)} & \dots & a_{nn} \end{vmatrix}}{\begin{vmatrix} A \end{vmatrix}}$$

Vandermonde Determinant

$$\begin{vmatrix} 1 & 1 & & 1 \\ x_1 & x_2 & \dots & x_n \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Vector Space

Vector Space Axioms

1. $\mathbf{u}, \mathbf{v} \in V \Rightarrow \mathbf{u} + \mathbf{v} \in V$	6. $\mathbf{u} \in V \Rightarrow k\mathbf{u} \in V$
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
4. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$	9. $k(m\mathbf{u}) = (km)\mathbf{u}$
5. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$	10. $1\mathbf{u} = \mathbf{u}$

Inner Product Axioms

Condition	$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$	$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^n$
Definition	$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \cdot \mathbf{v}$	$\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T \cdot \overline{\mathbf{v}}$
[Symmetry Property]	$\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$	$\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$
[Distrbutive Property]	$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$	$\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$
[Homogenity Property]	$\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$	$\langle k\mathbf{u}, \mathbf{v} \rangle = k \langle \mathbf{u}, \mathbf{v} \rangle$
[Positivity Property]	$\langle \mathbf{v}, \mathbf{v} \rangle \ge 0$	$\langle \mathbf{v}, \mathbf{v} \rangle \ge 0$

Dot Product - Euclidean Inner Product

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| cos \langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + \dots + u_n v_n$$

Cauchy-Schwarz Inequality

$$\mathbf{u}, \mathbf{v} \in R^n \Rightarrow |\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\| \|\mathbf{v}\|$$

Cross Product

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Subspace

Definition: W is a subspace of V

0. $W \subseteq V$
1. $\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$
2. $\mathbf{u} \in W \Rightarrow k\mathbf{u} \in W$

Linear Independence

Definition: $S = \{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\}$ is a **linearly independent** set

$$k_1 \mathbf{v_1} + k_2 \mathbf{v_2} + \dots + k_n \mathbf{v_n} = 0$$
$$\Leftrightarrow k_1 = k_2 = \dots = k_n = 0$$

Transition Matrix

$$B = \{\mathbf{u_1}, \mathbf{u_2}, \dots, \mathbf{u_n}\} \quad B' = \{\mathbf{u'_1}, \mathbf{u'_2}, \dots, \mathbf{u'_n}\}$$
$$P_{B \to B'} = \left[[\mathbf{u_1}]_{B'} \quad [\mathbf{u_2}]_{B'} \quad \dots \quad [\mathbf{u_n}]_{B'} \right] = (B')^{-1} B$$
$$[\mathbf{v}]_{B'} = P_{B \to B'} [\mathbf{v}]_B$$

Find Basis for span(S)

Suppose $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$

$$\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n}\end{bmatrix} \xrightarrow{\text{Gaussian Elimination}} \begin{bmatrix}\mathbf{v_1'} & \mathbf{v_2'} & \ldots & \mathbf{v_n'}\end{bmatrix}$$

$$\mathbf{v_i'} = \begin{bmatrix}\vdots \\ 0 \\ 1 \\ 0 \\ \vdots\end{bmatrix} \Rightarrow \text{all } \mathbf{v_i} \text{ form a basis for span}(S)$$

Dimension Formula

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

Linear Transformation

[Homogeneity property]	$T(k\mathbf{u}) = kT(\mathbf{u})$
[Additivity property]	$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

Isomorphism

Definition: T is an **isomorphism**, V and W are **isomorphic**
(If V and W are both n-dimensional, then V and W are isomorphic.)

$$T : V \to W \text{ is both one-to-one and onto}$$

Gram-Schmidt Process

1) Basis$\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}\}$ → orthogonal basis$\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$

$$\begin{aligned}
\mathbf{v_1} &= \mathbf{u_1} \\
\mathbf{v_2} &= \mathbf{u_2} - \frac{\langle \mathbf{u_2}, \mathbf{v_1}\rangle}{\|\mathbf{v_1}\|^2}\mathbf{v_1} \\
&\vdots \\
\mathbf{v_n} &= \mathbf{u_n} - \frac{\langle \mathbf{u_n}, \mathbf{v_1}\rangle}{\|\mathbf{v_1}\|^2}\mathbf{v_1} - \cdots - \frac{\langle \mathbf{u_n}, \mathbf{v_{n-1}}\rangle}{\|\mathbf{v_{n-1}}\|^2}\mathbf{v_{n-1}}
\end{aligned}$$

2) orthogonal basis$\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ $\xrightarrow{\text{normalize}}$ orthonormal basis$\{\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_n}\}$

Least Squares solution

$$A^T(A\mathbf{x} - \mathbf{b}) = 0 \xrightarrow{\text{minimize}} \|A\mathbf{x} - \mathbf{b}\|$$

Matrix Methods

Find Eigenvalues and Eigenvectors

Characteristic Polynomial	$p(\lambda) = \det(\lambda I - A)$
Eigenvalues	Solutions of $p(\lambda) = 0$
Eigenspace corresponding to λ_0	Solution Space of $(\lambda_0 I - A)\mathbf{x} = 0$

Diagonalization

Definition: A is diagonalizable

$$P^{-1}AP = D \text{ is diagonal}$$

Similarity

$B = P^{-1}AP$

A and B have the same **Determinant**, **Rank**, **Nullity**, **Trace**, **Invertibility, Characteristic Polynomial**, **Eigenvalues** and **Dimension of Eigenspace**

QR-Decomposition

Condition: $A = \begin{bmatrix}\mathbf{u_1} & \mathbf{u_2} & \ldots & \mathbf{u_n}\end{bmatrix}$ has full column rank

$$\{\mathbf{u_1}, \ldots, \mathbf{u_n}\} \xrightarrow{\text{Gram-Schmidt}} \{\mathbf{q_1}, \ldots, \mathbf{q_n}\}$$

$$A = QR = \begin{bmatrix}\mathbf{q_1} & \mathbf{q_2} & \ldots & \mathbf{q_n}\end{bmatrix}\begin{bmatrix}\langle \mathbf{u_1}, \mathbf{q_1}\rangle & \langle \mathbf{u_2}, \mathbf{q_1}\rangle & \ldots & \langle \mathbf{u_n}, \mathbf{q_1}\rangle \\ & \langle \mathbf{u_2}, \mathbf{q_2}\rangle & \ldots & \langle \mathbf{u_n}, \mathbf{q_2}\rangle \\ & & \ddots & \vdots \\ & & & \langle \mathbf{u_n}, \mathbf{q_n}\rangle\end{bmatrix}$$

Orthogonal Diagonalization

A is symmetric $\Rightarrow A = QDQ^T$, where Q is orthogonal and D is diagonal

$$\text{eigenspace } \lambda_1 : \text{basis} \{\mathbf{v_1}, \ldots, \mathbf{v_k}\} \xrightarrow{\text{Gram-Schmidt}} \{\mathbf{q_1}, \ldots, \mathbf{q_k}\}$$

$$\vdots$$

$$\text{eigenspace } \lambda_s : \text{basis} \{\mathbf{v_p}, \ldots, \mathbf{v_n}\} \xrightarrow{\text{Gram-Schmidt}} \{\mathbf{q_p}, \ldots, \mathbf{q_n}\}$$

$$Q = \begin{bmatrix} \mathbf{q_1} & \ldots & \mathbf{q_n} \end{bmatrix}, D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{bmatrix}$$

Singular Value Decomposition

$$A_{m\times n} = U\Sigma V^T = \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} & \ldots & \mathbf{u_m} \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}_{m\times n} \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \ldots & \mathbf{v_n} \end{bmatrix}^T$$

Singular values of A: $\sigma_1 = \sqrt{\lambda_1} \geq \cdots \geq \sigma_n = \sqrt{\lambda_n}$

$\lambda_1, \ldots, \lambda_n$ are eigenvalues of $A^T A$

$V = [\mathbf{v_1}, \ldots, \mathbf{v_n}]$ orthogonally diagonalizes $A^T A$

$$\mathbf{u_i} = \frac{A\mathbf{v_i}}{\sigma_i} \ (i = 1, \ldots, n)$$

Hermitian, Unitary, and Normal Matrices

For Real Matrices	For Complex Matrices
A^T [Transpose]	$A^* = \overline{A}^T$ [Conjugate Transpose]
$A = A^T$ [Symmetric]	$A = A^*$ [Hermitian]
$AA^T = I$ [Orthogonal]	$AA^* = I$ [Unitary]
$P^T AP = D$ [Orthogonally Diagonalizable]	$P^* AP = D$ [Unitarily Diagonalizable]
$A^T = -A$ [Skew-symmetric]	$A^* = \overline{A}^T = -A$ [Skew-Hermitian]
	$AA^* = A^* A$ [Normal]

Quadratic Form

$$x^T Ax = x^T PDP^T x = (P^T x)^T D(P^T x)$$

$x = Py$ is an orthogonal change of variable

Positive Definite, Semidefinite and Indefinite

positive definite	$\Leftrightarrow$	$x^T Ax > 0$ for $x \neq 0$
	$\Leftrightarrow$	A's eigenvalues $\lambda_1, \ldots, \lambda_n > 0$
	$\Leftrightarrow$	the determinant of A's principal submatrices are positive
negative definite	$\Leftrightarrow$	$x^T Ax < 0 \ for \ x \neq 0$
indefinite	$\Leftrightarrow$	has both positive and negative values
	$\Leftrightarrow$	$\exists \ \lambda_p > 0, \lambda_q < 0$
positive semidefinite	$\Leftrightarrow$	$x^T Ax \geq 0 \ for \ x \neq 0$
negative semidefinite	$\Leftrightarrow$	$x^T Ax \leq 0 \ for \ x \neq 0$